Discrete Mathematics 2025 Spring

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9.1.1 Binary and Unary Operations

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9.1.2 Properties of Binary Operations

- Commutativity, Associativity, Idempotency, Distributivity, Absorption.
- Special Elements: Identity Element, Zero Element, Invertible Element.
- Cancellation Laws

- **Definition 9.1:** Let S be a set. A function $f : S \times S \to S$ is called a *binary operation* on S, or simply a binary operation. We also say that S is **closed** under f .
- **Examples:**

(1) Binary operations on the set of natural numbers **N**: addition and multiplication.

Note: Subtraction and division do not satisfy closure, as the result may not be a natural number.

(2) Binary operations on the set of integers **Z**: addition, subtraction, and multiplication.

Note: Division may produce non-integer results, thus lacking closure.

(3) Binary operations on the set of nonzero real numbers **R***: multiplication and division.

Note: Addition and subtraction do not satisfy closure, as results can include zero.

- **Examples:**

(4) Let $S=\{ a_1, a_2, \ldots , a_n \}$, $a_i \circ a_j = a_i$, $\circ$ is a binary operation on S.

(5) Let $M_n(R)$ denote the set of all n ($n\geq 2$) real matrices, then

$$M_n(\mathrm{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \middle| \; a_{ij} \in \mathrm{R},\; i,j = 1,2,\ldots,n \right\}$$

Matrix addition and multiplication are both binary operations on $M_n(R)$.

(6) Binary operations on the power set $P(S)$: $\cup$、$\cap$、$-$、$\oplus$.

(7) S^S be the set of all functions from S, the *composition operation* $\circ$ is a binary operation on S^S.

↳ Unary operations and *n-ary* operations

- **Definition 9.2:** Let S be a set, A function $f: S \rightarrow S$ is called a ***unary operation*** on S, or simply a unary operation.
  - More generally, a function $f: \underbrace{S \times S \times \ldots \times S}_{n} \rightarrow S$ is called an ***n-ary operation*** on S, or simply an ***n*-operation**.
- **Examples:** The following are all unary operations:

  (1) The ***negation*** (taking the additive inverse) operation on **Z**, **Q** *and* **R** .

  (2) The ***reciprocal*** (taking the multiplicative inverse) operation on the set of nonzero rational numbers **Q*** and the set of nonzero real numbers **R***.

  (3) The ***complex conjugate*** operation on the set of complex numbers **C**.

  (4) On the power set $P(S)$ with universe S, the ***absolute complement*** operation~ .

  (5) Let A be the set of all bijective functions on S $A \subseteq S^S$, the ***inverse function*** operation.

  (6) **On M_n(R) ($n \geq 2$)** the ***matrix transpose*** operation.

- **Operators:** $\circ$, $*$, $\cdot$, $\oplus$, $\otimes$ are used to denote binary or unary operations.
  - For a ***binary operation*** $\circ$, if x and y operate to produce z, it is denoted as $x \circ y = z$.
  - For a *unary operation* $\circ$, the result of applying it to x is written as $\circ x$.
  - For an ***n-ary operation***, we write $\circ(a_1, a_2, \ldots, a_n) = b$.
- Ways to represent binary or unary operations：***formulas***、***operation tables***.

Note: In the same problem or context, different operations should be represented using different operator symbols.

Formula and Table Representations of Operations

Formula Representation

- **Example**： Let **R** be the set of real numbers. Define a binary operation **R as**： $\forall x, y \in R$, $x * y = x$.

  Such as: $3 * 4 = 3$, $0.5 * (-3) = 0.5$

Tables Representation： used to represent unary and binary operations on finite sets。

Operation Table for Unary Operations

	$\circ a_i$
a_1	$\circ a_1$
a_2	$\circ a_2$
…	…
a_n	$\circ a_n$

Operation Table for Binary Operations

$\circ$	a_1	a_2	…	a_n
a_1	$a_1 \circ a_1$	$a_1 \circ a_2$		$a_1 \circ a_n$
a_2	$a_2 \circ a_1$	$a_2 \circ a_2$		$a_2 \circ a_n$
…				
a_n	$a_n \circ a_1$	$a_n \circ a_2$		$a_n \circ a_n$

↳ Example of Operation Table Representation ions

- **Example:** Set **S= {*a*,*b*}**, Provide the operation tables $\oplus$ and $\sim$ on the power set ***P*(*S*)** .

$\oplus$Operation Table

$\oplus$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\varnothing$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\varnothing$	$\{a,b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\varnothing$	$\{a\}$
$\{a,b\}$	$\{a,b\}$	$\{b\}$	$\{a\}$	$\varnothing$

$\sim$Operation Table

X	$\sim X$
$\varnothing$	$\{a,b\}$
$\{a\}$	$\{b\}$
$\{b\}$	$\{a\}$
$\{a,b\}$	$\varnothing$

↳ Example of Operation Table Representation

- **Example:** Let Z_5= { 0, 1, 2, 3, 4 }, ⊕, ⊗ represent addition and multiplication modulo 5, respectively.

⊕Operation Table

⊕	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

⊗Operation Table

⊗	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

9.1 Binary Operations and Their Properties

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- **Definition 9.3:** Let $\circ$ be a binary operation on S ,

(1) If for all $x, y \in S$, $x \circ y = y \circ x$, then the operation is said to satisfy the ***commutative law*** (or is **commutative**) on S .

(2) If for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, then the operation is said to satisfy the ***associative law*** (or is **associative**) on S .

(3) If for all $x \in S$, $x \circ x = x$, then the operation is said to satisfy the ***idempotent law*** (or is **idempotent**) on S.

- **Example: Z, Q, R** denote the sets of integers, rational numbers, and real numbers, respectively. M_n(R) denotes the set of all n×n real matrices, n≥2, $P(B)$ power set , A^A set of all functions from A to A, |A|≥2.

Set	Operations	Commutative law	Associative law	Idempotent law
Z, Q, R	$+$	Yes	Yes	No
	$\times$	Yes	Yes	No
M_n(R)	$+$	Yes	Yes	No
	$\times$	No	Yes	No
$P(B)$	$\cup$	Yes	Yes	Yes
	$\cap$	Yes	Yes	Yes
	$-$	No	No	No
	$\oplus$	Yes	Yes	No
A^A	$\circ$	No	Yes	No

Distributive & Absorption Laws (Two Binary Operations)

- **Definition 9.4:** Let $\circ$ **and** $*$ be two distinct binary operations on S,

(1) If for all $x, y, z \in S$

$$(x * y) \circ z = (x \circ z) * (y \circ z)$$

$$z \circ (x * y) = (z \circ x) * (z \circ y)$$

then we say that the operation $\circ$ *distributes* over the operation $*$.

(2) If both $\circ$ and $*$ are commutative, and for all $x, y \in S$

$$x \circ (x * y) = x$$

$$x * (x \circ y) = x$$

then we say that the operations $\circ$ and $*$ satisfy the *absorption law*.

↳ Distributive & Absorption Laws (Two Binary Operations)(e.g.)

- **Example: Z, Q, R** denote the sets of integers, rational numbers, and real numbers, respectively; M_n(R) denotes the set of all $n \times n$ real matrices, $n \geq 2$, $P(B)$ power set , A^A set of all functions from A to A, $|A| \geq 2$.

Set	Operations	Distributive Law	Absorption Law
Z,Q,R	+ and ×	× to + distribute + to × Not distribute	No
M_n(R)	Matric + and matric ×	× to + distribute + to × Not distribute	No
$P(B)$	∪ and ∩	∪ to ∩ distribute ∩ to ∪ distribute	Yes
	∩ and ⊕	∩ to ⊕ distribute ⊕ to ∩ Not distribute	No